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On the mass transport induced by oscillatory flow in a turbulent boundary layer

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Oscillatory flow in a turbulent boundary layer is modelled by using a coefficient of eddy viscosity whose value depends upon distance from a fixed boundary. A general oscillatory flow is prescribed beyond the layer, and the model is used to calculate the mass transport velocity induced by this within the layer. The result is investigated numerically for a representative distribution of eddy viscosity and the conclusions interpreted in terms of the mass transport induced by progressive and standing waves. For progressive waves, the limiting value of the mass transport velocity at the outer edge of the layer is the same as for laminar flow. For standing waves, the limiting value is reduced relative to its laminar value but, within the lowermost 25 % of the layer, there is a drift which is reversed relative to the limiting value. This is considerably stronger than its counterpart in the laminar case and, in view of the greater thickness of the turbulent layer, it may make a dominant contribution to the net movement of loose bed material by a standing wave system.

1. Introduction

The existence of thin viscous boundary layers adjacent to boundaries in a vibrating fluid was shown by Longuet-Higgins (1953) to be of crucial importance in the determination of wave-induced mass transport. The possible relevance of these ideas in connexion with the wave-induced movement of suspended sediment has been pointed out by several authors. Amongst these, we may mention Abbott (1960), who applied the ideas to the movement of suspended material in the Thames Estuary, Russell & Osorio (1958), who conducted experiments to determine wave-induced drift in a closed channel and Hunt & Johns (1963), who extended some aspects of the analysis to three-dimensional situations in both rotating and non-rotating systems. Amongst more recent applications, Johns (1967, 1968) has used the ideas to determine the pattern of tidal-induced mass transport in river estuaries in which the turbulence in the tidal flow is parameterized in terms of a coefficient of eddy viscosity that represents a plausible momentum transfer process. In view of the fact that Longuet-Higgins's analysis, and all extensions thereof, are within the framework of laminar flow, the predictions of the present author must be regarded as being primarily of a qualitative nature. This point was fully emphasized in the author's before-mentioned publications. On the other hand, the experiments of Russell & Osorio (1958) and also Collins (1963) indicate that the flow in the boundary layer associated with gravity wave motion is likely to be turbulent in conditions appropriate to geophysical actuality. At the same time, the mass transport velocities observed by Russell & Osorio beneath a progressive wave system tended to be in agreement with the predictions of Longuet-Higgins's laminar theory. This circumstance led Longuet-Higgins, in a supplement to the work of Russell & Osorio (1958), to re-cast the analysis in terms of a coefficient of eddy viscosity whose value is dependent upon distance from the solid boundary—this being a first step towards an understanding of the phenomena in turbulent conditions. For progressive waves, the analysis reveals that the mass transport velocity at the outer edge of a turbulent boundary layer is independent of the functional form of the eddy viscosity. This result, then, yields a possible explanation of the observations made by Russell & Osorio. This conclusion, however, is not applicable to the case of a standing wave, and, beneath this, the induced mass transport may be modified in important respects by the existence of a turbulent boundary layer. In particular, we draw attention to the tidal flow in a river estuary wherein the tidal wave is rarely ever of a pure standing or progressive type. With such applications in mind, it is of some importance to inquire into the nature of the mass transport induced by a general oscillatory flow in a turbulent boundary layer.

In the present paper, we use the same parametric representation of the turbulence as suggested by Longuet-Higgins in the supplement to Russell & Osorio (1958). An appropriate analysis is carried through so as to obtain an expression for the mass transport velocity in the boundary layer induced by a general oscillatory flow beyond the layer. This involves the functional specification of the eddy viscosity. The formula is evaluated numerically for a specification that represents a plausible momentum transfer process. If the general result is then applied to the case of an oscillation resulting from a progressive wave, the mass transport velocity at the outer edge of the layer is found to be the same as in the laminar case—thus, in this example, confirming Longuet-Higgins's analytical argument. If, however, the oscillation results from a standing wave system, the mass transport velocity at the outer edge of the layer is much reduced relative to the laminar case. Within the boundary layer, there are other important differences which may lead to a different pattern of sediment transport.

2. Formulation

Conditions are referred to rectangular Cartesian axes Ox, Oz in which Ox is fixed horizontally within a fixed impermeable surface. The components of fluid velocity (suitably averaged so as to suppress the details of turbulent fluctuations) are denoted by u and w. The dominant component of the Reynolds stress τ in a boundary layer of thickness $O(\delta)$ is specified in terms of a coefficient of eddy viscosity and the instantaneous gradient of the averaged flow

$$\tau = -\nu\rho K(z)\,\partial u/\partial z. \tag{2.1}$$

The function $K(z)$ is specified in the boundary layer and will be chosen so as to represent the gross characteristics of a physically plausible momentum transfer process. Accordingly, the boundary-layer equation for the averaged motion takes the form

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\frac{\partial}{\partial z}\left(K\frac{\partial u}{\partial z}\right), \tag{2.2}$$

whilst for an incompressible fluid continuity of mass requires that

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z} = 0. \tag{2.3}$$

Additionally, we suppose that the horizontal pressure gradient within the boundary layer is equal to its value just beyond the layer where the mainstream velocity is $U(x,t)$. In (2.2), we therefore write

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}, \tag{2.4}$$

and apply the usual boundary conditions

$$u = w = 0 \quad \text{at} \quad z = 0 \tag{2.5}$$

and

$$u \sim U \quad \text{when} \quad z \gg \delta. \tag{2.6}$$

The mainstream is now taken to be a small-amplitude oscillatory flow and to be represented by the real part of

$$\epsilon U_1(x)\, e^{i\sigma t}, \tag{2.7}$$

where ϵ is a small ordering parameter. The horizontal velocity in the boundary layer is then expanded as a power series in ϵ in the form

$$u = \epsilon u_1+\epsilon^2 u_2+\dots. \tag{2.8}$$

Within the boundary layer, we introduce a non-dimensional variable η by writing

$$\eta = (\sigma/2\nu)^{\frac{1}{2}}\, z, \tag{2.9}$$

and deduce from (2.3) and (2.5) that

$$w = -\left(\frac{2\nu}{\sigma}\right)^{\frac{1}{2}}\int_0^{\eta}\frac{\partial u}{\partial x}d\eta. \tag{2.10}$$

By substitution in (2.2) from (2.4), (2.9) and (2.10), we then obtain

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}-\frac{\partial u}{\partial \eta}\int_0^{\eta}\frac{\partial u}{\partial x}d\eta = \frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+\tfrac{1}{2}\sigma\frac{\partial}{\partial \eta}\left(K\frac{\partial u}{\partial \eta}\right). \tag{2.11}$$

Since $U = O(\epsilon)$, the leading term in the expansion (2.8) satisfies

$$\frac{\partial u_1}{\partial t} = \frac{\partial U}{\partial t}+\tfrac{1}{2}\sigma\frac{\partial}{\partial \eta}\left(K\frac{\partial u_1}{\partial \eta}\right), \tag{2.12}$$

and by virtue of (2.7), we are led to propose a solution for u_1 in the form

$$u_1 = U_1(x)\,[1-F(\eta)]\,e^{i\sigma t}, \tag{2.13}$$

where only the real part is to be retained. The function F is found to satisfy

$$\frac{d}{d\eta}\left(K\frac{dF}{d\eta}\right)-2iF = 0, \tag{2.14}$$

whilst the boundary conditions (2.5) and (2.6) lead to

$$F(0) = 1; \quad F(\infty) = 0. \tag{2.15}$$

The second-order term in (2.8) has a time-independent component $\overline{u_2}$ in addition to an oscillatory component. The former of these satisfies

$$\overline{u_1 \frac{\partial u_1}{\partial x}} - \overline{\frac{\partial u_1}{\partial \eta} \int_0^\eta \frac{\partial u_1}{\partial x} d\eta} = \overline{U \frac{\partial U}{\partial x}} + \tfrac{1}{2}\sigma \frac{\partial}{\partial \eta}\left(K \frac{\partial \overline{u_2}}{\partial \eta}\right), \tag{2.16}$$

where the horizontal overbar denotes the mean value. The condition of no slippage at the surface leads to

$$\overline{u_2} = 0 \quad \text{at} \quad \eta = 0 \tag{2.17}$$

but, as has been pointed out by Stuart (1966) and Riley (1965), it is not possible to make $\overline{u_2} = 0$ at $\eta = \infty$ within the framework of the single boundary-layer structure considered in this paper. The best that we can do is to satisfy the condition

$$\overline{u_2} \quad \text{finite at} \quad \eta = \infty. \tag{2.18}$$

This would then form the boundary condition at the inner edge of a second boundary layer in which $\overline{u_2}$ decays to zero.

Bearing in mind that only the real part of u_1 has significance, the introduction of (2.13) into (2.16) reveals that $\overline{u_2}$ is given by

$$\sigma \frac{\partial}{\partial \eta}\left(K \frac{\partial \overline{u_2}}{\partial \eta}\right) = U_1^*(x) \frac{dU_1}{dx}\left[|F(\eta)|^2 - F(\eta) - F^*(\eta) + \frac{dF^*}{d\eta}\left\{\eta - \int_0^\eta F(\eta)\, d\eta\right\}\right], \tag{2.19}$$

where the asterisk denotes the complex conjugate and only the real part of $\overline{u_2}$ is to be retained. We therefore write a solution for $\overline{u_2}$ in the form

$$\overline{u_2} = \frac{1}{\sigma} U_1^*(x) \frac{dU_1}{dx} G(\eta), \tag{2.20}$$

where
$$\frac{d}{d\eta}\left(K \frac{dG}{d\eta}\right) = |F(\eta)|^2 - F(\eta) - F^*(\eta) + \frac{dF^*}{d\eta}\left\{\eta - \int_0^\eta F(\eta)\, d\eta\right\}, \tag{2.21}$$

subject to
$$G(0) = 0; \quad G(\eta) \quad \text{finite as} \quad \eta \to \infty. \tag{2.22}$$

3. Solution of equations

For a specified functional form of K, the solution of (2.14) subject to (2.15) can, in general, only be obtained by numerical integration. On the other hand, the solution of (2.21) subject to (2.22) can be expressed in terms of relatively simple quadratures involving F. This we now proceed to do. From (2.14), we see that

$$2i \int_0^\eta F\, d\eta = \left[K \frac{dF}{d\eta}\right]_0^\eta,$$

and on defining $K = 1$ at $\eta = 0$ this yields

$$\int_0^\eta F\, d\eta = \tfrac{1}{2}i\left[F'(0) - K \frac{dF}{d\eta}\right]. \tag{3.1}$$

Hence, (2.21) may be written

$$\frac{d}{d\eta}\left(K\frac{dG}{d\eta}\right) = |F|^2 - 2\operatorname{Re}F + \eta\frac{dF^*}{d\eta} + \tfrac{1}{2}i\left[K\left|\frac{dF}{d\eta}\right|^2 - F'(0)\frac{dF^*}{d\eta}\right]. \tag{3.2}$$

This equation may now be integrated from $\eta = 0$ to η by using the boundary conditions (2.15) together with a further application of (3.1). This procedure yields

$$K\frac{dG}{d\eta} = C + \int_0^\eta |F|^2 d\eta - \operatorname{Im}\left\{K\frac{dF}{d\eta} - F'(0)\right\} + \int_0^\eta \eta\frac{dF^*}{d\eta}d\eta + \tfrac{1}{2}i\int_0^\eta K\left|\frac{dF}{d\eta}\right|^2 d\eta - \tfrac{1}{2}iF'(0)\{F^* - 1\}, \tag{3.3}$$

where C is a constant of integration. Both the integrals in (3.3) may be evaluated explicitly in terms of F, thus yielding an expression for $dG/d\eta$ free from quadratures. An integration by parts gives

$$\int_0^\eta K\left|\frac{dF}{d\eta}\right|^2 d\eta = \int_0^\eta K\frac{dF}{d\eta}\frac{dF^*}{d\eta}d\eta = \left[K\frac{dF}{d\eta}F^*\right]_0^\eta - \int_0^\eta F^*\frac{d}{d\eta}\left\{K\frac{dF}{d\eta}\right\}d\eta. \tag{3.4}$$

Hence, by use of (2.14) and (2.15), it follows that

$$\int_0^\eta K\left|\frac{dF}{d\eta}\right|^2 d\eta = K\frac{dF}{d\eta}F^* - F'(0) - 2i\int_0^\eta |F|^2 d\eta. \tag{3.5}$$

On separately equating real and imaginary parts in (3.5), we find that

$$\int_0^\eta K\left|\frac{dF}{d\eta}\right|^2 d\eta = \operatorname{Re}\left\{K\frac{dF}{d\eta}F^* - F'(0)\right\} \tag{3.6}$$

and

$$\int_0^\eta |F|^2 d\eta = \tfrac{1}{2}\operatorname{Im}\left\{K\frac{dF}{d\eta}F^* - F'(0)\right\}. \tag{3.7}$$

The insertion of these expressions in (3.3) then leads to

$$K\frac{dG}{d\eta} = C - \operatorname{Im}\left\{(1-F^*)K\frac{dF}{d\eta}\right\} + \eta F^* - \tfrac{1}{2}i\left[K\frac{dF^*}{d\eta} - \{F'(0)\}^*\right] + \tfrac{1}{2}i\left[K\frac{dF}{d\eta}F^* - F'(0)F^*\right]. \tag{3.8}$$

We now choose C so as to ensure that G remains finite as $\eta \to \infty$. Since $F \to 0$ as $\eta \to \infty$, an infinite value of G can only result from the terms independent of η on the right-hand side of (3.8). The occurrence of such terms may be excluded by choosing

$$C = -\tfrac{1}{2}i\{F'(0)\}^*, \tag{3.9}$$

in which case

$$K\frac{dG}{d\eta} = \operatorname{Im}\left\{K\frac{dF}{d\eta}F^* - K\frac{dF}{d\eta}\right\} + \eta F^* - \tfrac{1}{2}i\,K\frac{dF^*}{d\eta} + \tfrac{1}{2}i\,K\frac{dF}{d\eta}F^* - \tfrac{1}{2}iF'(0)F^*. \tag{3.10}$$

An integration of (3.10) subject to $G(0) = 0$ therefore yields

$$G = \operatorname{Im}\left[\int_0^\eta\left\{F^*\frac{dF}{d\eta} - \frac{dF}{d\eta}\right\}d\eta\right] + \tfrac{1}{2}i\int_0^\eta\left\{F^*\frac{dF}{d\eta} - \frac{dF^*}{d\eta}\right\}d\eta + \int_0^\eta\frac{\{\eta - \tfrac{1}{2}iF'(0)\}F^*}{K}d\eta. \tag{3.11}$$

Finally, algebraic manipulation of (3.11) leads to the result

$$G = \tfrac{1}{2}i\{|F|^2+F-2F^*\}-\tfrac{1}{2}i\int_0^\eta \left[F^*\frac{dF}{d\eta}+\frac{F^*\{2i\eta+F'(0)\}}{K}\right]d\eta. \tag{3.12}$$

In particular, as $\eta \to \infty$,

$$G \sim -\tfrac{1}{2}i\int_0^\eta F^*\left[\frac{dF}{d\eta}+\frac{2i\eta+F'(0)}{K}\right]d\eta. \tag{3.13}$$

We may now use (3.12) to compute the mass transport, or Lagrangian drift velocity, within the boundary layer. On using the well-known result of Longuet-Higgins (1953), this is found to be given by the real part of

$$\bar{U} = \frac{1}{\sigma}H(\eta)\,U_1^*\frac{dU_1}{dx}, \tag{3.14}$$

where

$$H(\eta) = G(\eta)+\tfrac{1}{4}K\left|\frac{dF}{d\eta}\right|^2-\tfrac{1}{4}F'(0)\frac{dF^*}{d\eta}+\tfrac{1}{2}i\left\{1-2\,\mathrm{Re}\,F+|F|^2-\eta\frac{dF^*}{d\eta}\right\}, \tag{3.15}$$

and, as $\eta \to \infty$,

$$H \sim \tfrac{1}{2}i+G(\infty). \tag{3.16}$$

4. Numerical solution

In order to proceed to a numerical evaluation of the formulae in §3, we must specify the functional form of K. In the present paper, this is chosen so as to model a distribution of turbulence, the intensity of which increases outwards from the fixed boundary. The physical principle upon which the choice is made is simply that turbulence tends to be inhibited on approaching the surface. Accordingly, we take a representative distribution to be given by

$$K = K_\infty+(1-K_\infty)\,e^{-\eta}. \tag{4.1}$$

With this specification, we see that $K = 1$ at $\eta = 0$ (as required earlier), and $K = K_\infty$ as $\eta \to \infty$. With a view to obtaining a numerical solution of (2.14) subject to (2.15), it is necessary to apply the outer boundary condition at a suitably large (but finite) value of η (say η_1), and to replace the differential equation by a sequence of difference equations. On using central difference representations of the derivatives, and writing

$$\eta = j\delta \quad (j = 0,1,2,\ldots,n; \quad n\delta = \eta_1),$$

these are found to have the form

$$\frac{K_j[F_{j+1}-2F_j+F_{j-1}]}{\delta^2}+\frac{K_j'[F_{j+1}-F_{j-1}]}{2\delta}-2iF_j = 0 \quad (j = 1,2,\ldots,n-1). \tag{4.2}$$

From (2.15), the boundary conditions which complete this set are

$$F_0 = 1, \quad F_n = 0. \tag{4.3}$$

In the present numerical evaluations, (4.2) and (4.3) have been solved by an iterative technique starting from an initial guessed solution. Having thus computed F at a discrete sequence of points, the definite integral of any expression involving F may be calculated by application of the trapezoidal rule. This scheme has been programmed for the computation of the functions F and H with K given by (4.1), the finite step length being decreased until two successive approximations are within a specified tolerance.

In these numerical computations, the parameter K_∞ was chosen to have the values 1·0 (with $\eta_1 = 10$), corresponding to the laminar case, and 100·0 (with $\eta_1 = 60$) corresponding to the turbulent case. This latter value of K_∞, when applied to oscillations of a tidal scale, is suggested by calculations undertaken by the author of the flow in a tidal estuary (Johns 1966, 1967). The variation of the function F is shown in figure 1, the inset representing the laminar case.

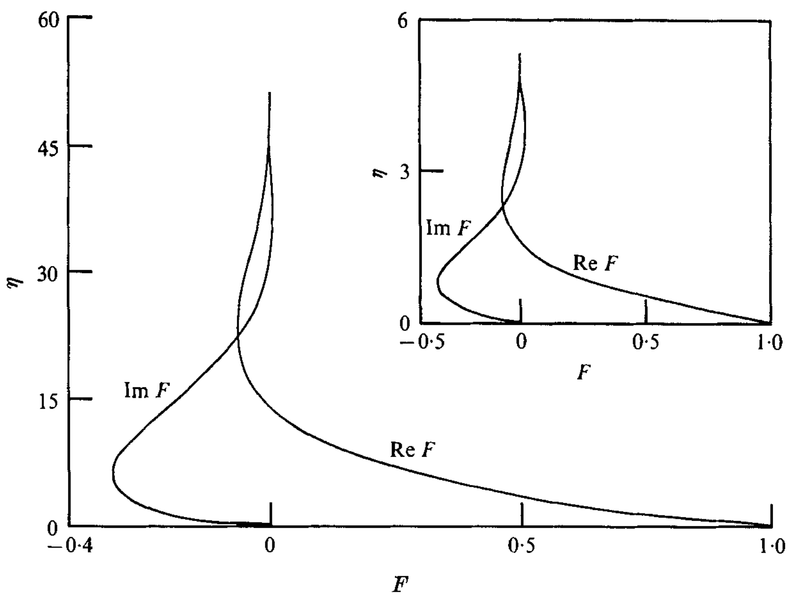

FIGURE 1. Variation of F with η in laminar and turbulent cases.

Inspection reveals that the choices of the parameter η_1 are satisfactory in view of the fact that the outer boundary conditions are effectively satisfied for $\eta < \eta_1$. For the laminar case, the boundary-layer thickness is about $5(2\nu/\sigma)^{\frac{1}{2}}$ whilst for the turbulent case it is about $50(2\nu/\sigma)^{\frac{1}{2}}$. The overall profiles of F in the two cases are very similar.

The variation of H with η is shown in figure 2, the inset again representing the laminar case. With a view to intepreting this in terms of the mass transport velocity induced by an oscillatory wave motion, it is convenient to specify first a progressive wave for which the surface displacement is given by

$$\zeta = a\cos(kx + \sigma t), \tag{4.4}$$

and secondly a standing wave for which

$$\zeta = a\cos kx \cos\sigma t. \tag{4.5}$$

The velocity potentials corresponding to these are respectively

$$\phi = -\frac{a\sigma}{k}\frac{\cosh k(z+h)}{\sinh kh}\sin(kx+\sigma t) \tag{4.6}$$

and
$$\phi = -\frac{a\sigma}{k}\frac{\cosh k(z+h)}{\sinh kh}\cos kx \sin \sigma t, \tag{4.7}$$

with associated values of the bottom velocity obtainable from

$$U_1(x) = -\frac{a\sigma}{\sinh kh}e^{ikx} \tag{4.8}$$

and
$$U_1(x) = -\frac{ai\sigma}{\sinh kh}\sin kx. \tag{4.9}$$

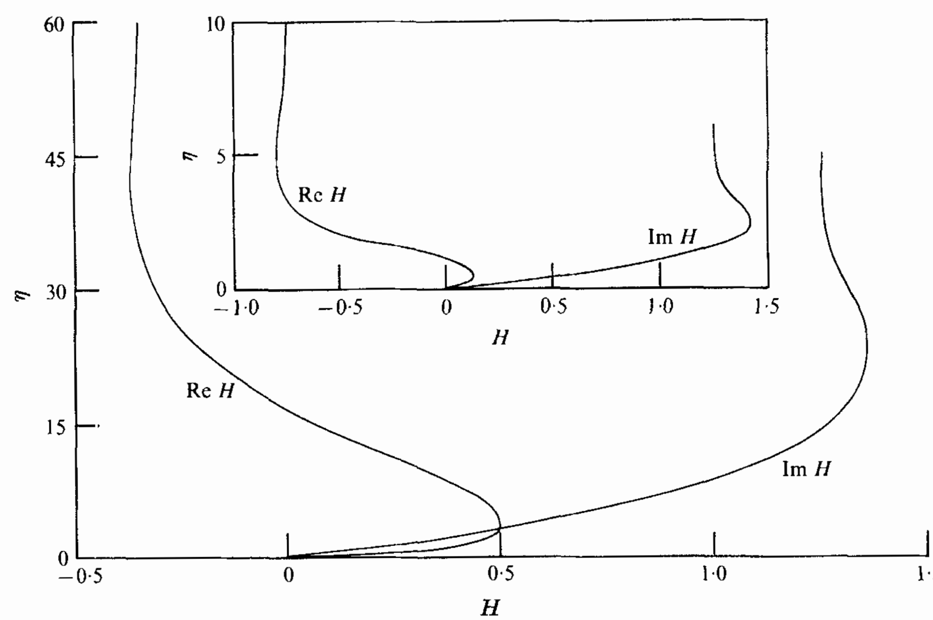

FIGURE 2. Variation of H with η in laminar and turbulent cases.

On using the formula (3.14), and taking the real part, we find that the associated Lagrangian drift velocities are given by

$$\overline{U} = -\frac{a^2k\sigma}{\sinh^2 kh}\operatorname{Im} H(\eta) \tag{4.10}$$

and
$$\overline{U} = \frac{a^2k\sigma}{2\sinh^2 kh}\sin 2kx \operatorname{Re} H(\eta). \tag{4.11}$$

These results, together with the numerical values of H in figure 2, have several interesting implications. We see that in both the laminar and turbulent cases

$$\operatorname{Im} H(\eta) \sim 1{\cdot}25, \tag{4.12}$$

for sufficiently large values of η. In other words, the mass transport velocity induced at the outer edge of the layer by a progressive wave is the same in both cases. This is in accord with the arguments advanced in this connexion by Longuet-Higgins in the supplement to Russell & Osorio (1958). It should also be noted that, within the boundary layer, the maximum drift velocity is slightly greater in the laminar case than in the turbulent case.

For a standing wave, the situation at the outer edge of the layer is significantly different. In the laminar case, the limiting value of $\mathrm{Re}\, H(\eta)$ is $-0{\cdot}75$, whilst in the turbulent case it is approximately $-0{\cdot}34$. In other words, the mass transport velocity induced at the outer edge of the layer by a standing wave is substantially less in the turbulent case than in the laminar case. At the same time, the turbulent boundary layer leads to a strong drift within the lowermost 25 % of the layer in a direction that is opposite to its limiting value. This is in marked contrast to the laminar case, where the corresponding drift takes place within the lowermost 10 % of the layer, and is about 80 % less as regards maximum magnitude.

From the point of view of the application of these results to the determination of the sediment transport induced by a wave motion, it should be observed that the thickness of the turbulent layer is much greater than that of the laminar layer. In the laminar case, a suspension of loose bed material will probably be present at the outer edge of the layer, and the drift velocity at this level will probably give an indication of the direction and magnitude of the sediment transport. In the turbulent case, the greater thickness of the layer is quite likely to be such that there will be little sediment in suspension at the outer edge. Any suspension that does exist will be densest in the lowermost parts of the layer adjacent to the fixed boundary. Accordingly, on general grounds, we might expect a progressive wave motion to transport less suspended material in the turbulent boundary layer than in the laminar layer. A similar remark applies to a standing wave system. In the turbulent layer, the existence of the strong drift within the lowermost 25 % of that layer will probably be the dominant mechanism controlling the movement of loose bed material. It is possible, then, that the direction of the net sediment movement beneath a standing wave will be different in the two cases.

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